問題

[1] 微分

(1) $y = \sqrt[3]{x^3+1}$

(2) $y = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

(3) $y = xe^{1-x}$

(4) $y = \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

(5) $y = e^{-2x}\cos 3x$

(6) $y = \sqrt{2x}\log x$

(7) $y = \sin^3 x$

(8) $y = \log\left(\dfrac{1-x}{1+x}\right)$

(9) $y = \sin^{-1} 2x$

(10) $y = \log\left(\dfrac{1-\cos x}{1+\cos x}\right)$

[2] 積分

(1) $I = \displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

(2) $I = \displaystyle\int x(x^2+1)^9 dx$

(3) $I = \displaystyle\int x\sqrt{x+1}dx$

(4) $I = \displaystyle\int \log x dx$

(5) $I = \displaystyle\int \frac{x}{x^2-1}dx$

(6) $I = \displaystyle\int x^2 e^x dx$

(7) $I = \displaystyle\int \frac{dx}{x^2+3x+2}$

(8) $I = \displaystyle\int \frac{2x+3}{x^2+3x+2}dx$

(9) $I = \displaystyle\int e^x \cos x dx$

(10) $I = \displaystyle\int \cos^2 x dx$

解答

[1] 微分

(1) $y = \sqrt[3]{x^3+1}$

$$\begin{aligned} y' &= \frac{d}{dx}(x^3+1)^{\frac{1}{3}} \\ &= \frac{1}{3}(x^3+1)^{-\frac{2}{3}} \times (x^3+1)' \\ &= \frac{1}{3}(x^3+1)^{-\frac{2}{3}} \times 3x^2 \\ &= \frac{3x^2}{(x^3+1)^{\frac{2}{3}}} \end{aligned}$$

合成関数の微分

$$\begin{aligned} y &= f(t), \qquad \text{ただし、} t = g(x) \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \left\{\frac{d}{dt}f(t)\right\} \times \left\{\frac{d}{dx}g(x)\right\} \\ &= f'(t)g'(x) \end{aligned}$$

注意

$$\left(\sqrt[3]{x^3+1}\right)' = \frac{1}{3}(x^3+1)^{-\frac{2}{3}}\underline{\times (x^3+1)'}$$

$(x^3+1)'$ を掛けるのを忘れないこと。

(2) $y = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

$t = \sqrt{x}$ と置いて、

$$\frac{dy}{dx} = \left\{\frac{t-1}{t+1}\right\}' \times \left(\sqrt{x}\right)'$$

$$
\begin{aligned}
&= \frac{(t-1)'(t+1)-(t-1)(t+1)'}{(t+1)^2} \times \frac{1}{2\sqrt{x}} \\
&= \frac{2}{(t+1)^2} \times \frac{1}{2\sqrt{x}} \\
&= \frac{1}{(\sqrt{x}+1)^2\sqrt{x}}
\end{aligned}
$$

注意　商の微分公式

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

記憶に自身がない場合は、$f = x, g = 1$、$f = 1, g = x$ など簡単な例で確認する。

(3) $y = xe^{1-x}$

$$
\begin{aligned}
y' &= (x)' \times e^{1-x} + x \times \left(e^{1-x}\right)' \\
&= 1 \times e^{1-x} + x \times \left\{e^{1-x}(1-x)'\right\} \\
&= e^{1-x} + x \times e^{1-x} \times (-1) \\
&= (1-x)e^{1-x}
\end{aligned}
$$

(4) $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

分母、分子に e^x をかけ算して、整理すると、

$$y = \frac{e^{2x}+1}{e^{2x}-1}$$

ここで、$t = e^{2x}$ と置くと、

$$
\begin{aligned}
y &= \frac{t+1}{t-1} \\
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\
&= \left\{\frac{t+1}{t-1}\right\}' \cdot \left(e^{2x}\right)' \\
&= \frac{(t+1)'(t-1)-(t+1)(t-1)'}{(t-1)^2} \times (2e^{2x}) \\
&= -\frac{4e^{2x}}{(e^{2x}-1)^2}
\end{aligned}
$$

(5) $y = e^{-2x}\cos 3x$

$$
\begin{aligned}
y' &= \left(e^{-2x}\right)' \cos 3x + e^{-2x}\left(\cos 3x\right)' \\
&= e^{-2x}(-2x)' \cos 3x + e^{-2x}(-\sin 3x) \times (3x)' \\
&= -2e^{-2x}\cos 3x + e^{-2x}(-3\sin 3x) \\
&= -e^{-2x}(2\cos 3x + 3\sin 3x)
\end{aligned}
$$

(6) $y = \sqrt{2x}\log x$

$$
\begin{aligned}
y' &= \left(\sqrt{2x}\right)' \log x + \sqrt{2x}(\log x)' \\
&= \sqrt{2} \times \left(x^{\frac{1}{2}}\right)' \log x + \sqrt{2x} \times \frac{1}{x} \\
&= \frac{\sqrt{2}}{2\sqrt{x}} \times \log x + \frac{\sqrt{2x}}{x} \\
&= \frac{\log x}{\sqrt{2x}} + \frac{\sqrt{2}}{\sqrt{x}} \\
&= \frac{2+\log x}{\sqrt{2x}}
\end{aligned}
$$

(7) $y = \sin^3 x$

$$
\begin{aligned}
y' &= \left\{(\sin x)^3\right\}' \\
&= 3(\sin x)^2 \times (\sin x)' \\
&= 3\sin^2 x \cos x
\end{aligned}
$$

(8) $y = \log\left(\dfrac{1-x}{1+x}\right)$

$$
\begin{aligned}
y &= \log(1-x) - \log(1+x) \\
y' &= \frac{(1-x)'}{1-x} - \frac{(1+x)'}{1+x} \\
&= \frac{-1}{1-x} - \frac{1}{1+x} \\
&= -\frac{(1+x)+(1-x)}{(1-x)(1+x)} \\
&= -\frac{2}{1-x^2}
\end{aligned}
$$

(9) $y = \sin^{-1} 2x$

$2x = \sin y$ であるから、

$$
\begin{aligned}
\frac{dx}{dy} &= \frac{d}{dy}\left\{\frac{1}{2}\sin y\right\} \\
&= \frac{1}{2}\cos y \\
\frac{dy}{dx} &= \frac{2}{\cos y}
\end{aligned}
$$

右辺に含まれる y を $\sin y = 2x$ の関係を利用して x を用いて表す。

$$\frac{dy}{dx} = \pm\frac{2}{\sqrt{1-\sin^2 y}}$$

$$= \pm\frac{2}{\sqrt{1-(2x)^2}}$$
$$= \pm\frac{2}{\sqrt{1-4x^2}}$$

(10) $y = \log\left(\frac{1-\cos x}{1+\cos x}\right)$

$t = \cos x$ と置くと、

$$\begin{aligned}
y &= \log\left(\frac{1-t}{1+t}\right) \\
&= \log(1-t) - \log(1+t) \\
\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\
&= \{\log(1-t) - \log(1+t)\}' \times (\cos x)' \\
&= \left\{\frac{(1-t)'}{(1-t} - \frac{(1+t)'}{1+t}\right\} \times (-\sin x) \\
&= \frac{(1+t)+(1-t)}{(1-t)(1+t)} \times \sin x \\
&= \frac{2\sin x}{(1-\cos^x)(1+\cos x)} = \frac{2\sin x}{1-\cos^2 x} \\
&= \frac{2\sin x}{\sin^2 x} = \frac{2}{\sin x}
\end{aligned}$$

[2] 積分

(1) $I = \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$$\begin{aligned}
&(\sqrt{x} - \frac{1}{\sqrt{x}})^2 \\
&= (\sqrt{})^2 - 2\sqrt{x} \times \frac{1}{\sqrt{x}} + \left(\frac{1}{\sqrt{x}}\right)^2 \\
&= x - 2 + \frac{1}{x}
\end{aligned}$$

であるから、

$$\begin{aligned}
I &= \int (x - 2 + \frac{1}{x}) dx \\
&= \frac{1}{2}x^2 - 2x + \log x + C
\end{aligned}$$

ここで、C は積分定数である。

(2) $I = \int x(x^2+1)^9 dx$

$(x^2+1)^9$ を展開するのは厄介であるから、ここを変数変換し展開の必要をなくす。

$t = x^2+1$ と置くと、$dt = (x^2+1)'dx = 2xdx$ であるから、

$$\begin{aligned}
I &= \int \underbrace{(x^2+1)^9}_{t^9} \times \underbrace{xdx}_{\frac{1}{2}dt} \\
&= \frac{1}{2}\int t^9 dt \\
&= \frac{1}{2} \times \frac{1}{10}t^1 0 + C \\
&= \frac{1}{20}(x^2+1)^{10} + C
\end{aligned}$$

(3) $I = \int x\sqrt{x+1}dx$

$\sqrt{x+1}$ よりも $\sqrt{t}$ のほうが積分が容易であるので、

$$t = x+1$$

と変換する。このとき、$dt = dx$ であるから、

$$\begin{aligned}
I &= \int \underbrace{x}_{t-1} \times \underbrace{\sqrt{x+1}}_{\sqrt{t}} \underbrace{dx}_{dt} \\
&= \int (t-1)\sqrt{t}dt = \int (t-1)t^{\frac{1}{2}} dt \\
&= \int (t^{\frac{3}{2}} - t^{\frac{1}{2}})dt \\
&= \frac{1}{\frac{3}{2}+1}t^{\frac{3}{2}+1} - \frac{1}{\frac{1}{2}+1}t^{\frac{1}{2}+1} + C \\
&= \frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}} + C \\
&= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C \\
&= \frac{2}{15}(x+1)^{\frac{3}{2}}\{3(x+1) - 5\} + C \\
&= \frac{2}{15}(3x-2)(x+1)^{\frac{3}{2}} + C
\end{aligned}$$

[別解] 部分積分

$$\int \underbrace{x}_{f} \times g'(x)dx = \underbrace{x}_{f} \times g(x) - \int \underbrace{1}_{f'} \times g(x)dx$$

と変形すれば $g(x)$ の積分に帰着することに着目して、部分積分を行う。

$$\begin{aligned}
I &= \int \underbrace{x}_{f} \times \underbrace{\sqrt{x+1}}_{g'} dx \\
&= \int \underbrace{x}_{f} \times \underbrace{\left\{\frac{2}{3}(x+1)^{\frac{3}{2}}\right\}'}_{g'} dx \\
&= \underbrace{x}_{f} \times \underbrace{\frac{2}{3}(x+1)^{\frac{3}{2}}}_{g} - \int \underbrace{1}_{f'} \times \underbrace{\frac{2}{3}(x+1)^{\frac{3}{2}}}_{g} dx \\
&= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}} dx \\
&= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3} \times \frac{2}{5}(x+1)^{\frac{5}{2}} + C \\
&= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{15}(x+1)^{\frac{3}{2}}\left\{5x-2(x+1)\right\}+C \\
&= \frac{2}{15}(3x-2)(x+1)^{\frac{3}{2}}+C
\end{aligned}
$$

(4) $I = \int \log x dx$

$$
\begin{aligned}
I &= \int \underbrace{(x)'}_{f'} \times \underbrace{\log x}_{g} dx \\
&= \underbrace{x}_{f} \times \underbrace{\log x}_{g} - \int \underbrace{x}_{f} \times \underbrace{\frac{1}{x}}_{g'} dx \\
&= x\log x - \int 1dx \\
&= x\log x - x + C
\end{aligned}
$$

[別解] 変数変換

$t = \log x$ と置き換えると、

$$
x = e^t, \qquad dx = (e^t)dt = e^t dt
$$

であるから、

$$
\begin{aligned}
I &= \int \underbrace{\log x}_{t} \underbrace{dx}_{e^t dt} \\
&= \int \underbrace{t}_{f} \underbrace{e^t}_{g'} dt \qquad \text{(部分積分)} \\
&= \underbrace{t}_{f} \underbrace{e^t}_{g} - \int \underbrace{1}_{f'} \underbrace{e^t}_{g} dt \\
&= te^t - \int e^t dt \\
&= te^t - e^t + C \ = (t-1)e^t + C \\
&= (\log x - 1) \times x + C
\end{aligned}
$$

(5) $I = \int \frac{x}{x^2-1}dx$

$t = x^2 - 1$ と置くと、$dt = (x^2-1)'dx = 2xdx$ であるから、

$$
\begin{aligned}
I &= \int \underbrace{\frac{1}{x^2-1}}_{t} \times \underbrace{xdx}_{\frac{1}{2}dt} \\
&= \frac{1}{2}\int \frac{dt}{t} \\
&= \frac{1}{2}\log t + C \\
&= \frac{1}{2}\log(x^2-1) + C
\end{aligned}
$$

(6) $I = \int x^2 e^x dx$

部分積分により、

$$
\begin{aligned}
I &= \int \underbrace{x^2}_{f} \times \underbrace{e^x}_{g'} dx \\
&= \underbrace{x^2}_{f} \times \underbrace{e^x}_{g} - \int \underbrace{2x}_{f'} \times \underbrace{e^x}_{g} dx \\
&= x^2 e^x - 2\int \underbrace{x}_{f_2} \times \underbrace{e^x}_{g_2'} dx \\
&= x^2 e^x - 2\left\{ xe^x - \int \underbrace{1}_{f_2'} \times \underbrace{e^x}_{g_2} dx \right\} \\
&= x^2 e^x - 2\left\{xe^x - e^x\right\} + C \\
&= (x^2 - 2x + 2)e^x + C
\end{aligned}
$$

(7) $I = \int \frac{dx}{x^2+3x+2}$

部分分数展開

$$
\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}
$$

と展開すると、

$$
\begin{aligned}
A &= \left.\frac{1}{x+2}\right|_{x=-1} = 1 \\
B &= \left.\frac{1}{x+1}\right|_{x=-2} = -1
\end{aligned}
$$

であるから、

$$
\begin{aligned}
I &= \int \left\{ \frac{1}{x+1} - \frac{1}{x+2} \right\} dx \\
&= \log(x+1) - \log(x+2) + C \\
&= \log\frac{x+1}{x+2} + C
\end{aligned}
$$

(8) $I = \int \frac{2x+3}{x^2+3x+2}dx$

$t = x^2 + 3x + 2$ と置くと、

$$
dt = (x^2+3x+2)'dx = (2x+3)dx
$$

であるから、

$$
\begin{aligned}
I &= \int \frac{dt}{t} = \log t + C \\
&= \log(x^2+3x+2) + C
\end{aligned}
$$

[別解] 部分分数展開

$$
\frac{2x+3}{x^2+3x+2} = \frac{2x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}
$$

と展開すると、

$$\begin{aligned} A &= \left.\frac{2x+3}{x+2}\right|_{x=-1} = 1 \\ B &= \left.\frac{2x+3}{x+1}\right|_{x=-2} = 1 \end{aligned}$$

であるから、

$$\begin{aligned} I &= \int \left\{ \frac{1}{x+1} + \frac{1}{x+2} \right\} dx \\ &= \log(x+1) + \log(x+2) + C \\ &= \log\{(x+1)(x+2)\} + C \\ &= \log(x^2+3x+2) + C \end{aligned}$$

(9) $I = \int e^x \cos x dx$

$$I = \mathrm{Re}\left\{ \int e^x(\cos x + j\sin x)dx \right\}$$

であり、

$$\begin{aligned} &\int e^x(\cos x + j\sin x)dx \\ &= \int e^x e^{jx} dx \\ &= \int e^{(1+j)x} dx \\ &= \frac{1}{1+j} e^{(1+j)x} + C \\ &= \frac{1}{2}(1-j)e^x(\cos x + j\sin x) + C \\ &= \frac{1}{2}e^x(1-j)(\cos x + j\sin x) + C \end{aligned}$$

したがって、

$$\begin{aligned} I &= \frac{1}{2}e^x \mathrm{Re}\left[(1-j)(\cos x + j\sin x)\right] + C \\ &= \frac{1}{2}e^x(cosx + \sin x) + C \end{aligned}$$

[別解] 部分積分のくり返し

$$\begin{aligned} I &= \int \underbrace{e^x}_{f_1'} \times \underbrace{\cos x}_{g_1} dx \\ &= \underbrace{e^x}_{f_1} \times \underbrace{\cos x}_{g_1} - \int \underbrace{e^x}_{f_1} \times \underbrace{(-\sin x)_{g_1'}}\, dx \\ &= e^x\cos x + \int \underbrace{e^x}_{f_2'} \times \underbrace{\sin x}_{g_2} dx \\ &= e^x\cos x + e^x\sin x - \int \underbrace{e^x}_{f_2} \times \underbrace{\cos x}_{g_2'} dx \\ &= e^x\cos x + e^x\sin x - I \\ 2I &= e^x(\cos x + \sin x) \end{aligned}$$

以上より、

$$I = \frac{1}{2}e^x(\cos x + \sin x) + C$$

(10) $I = \int \cos^2 x dx$

$$\begin{aligned} I &= \int \frac{1}{2}(1+\cos 2x)dx \\ &= \frac{1}{2}\left\{ x + \frac{1}{2}\sin 2x \right\} + C \\ &= \frac{1}{4}\left\{2x + \sin 2x\right\} + C \end{aligned}$$

[別解] 部分積分

$$\begin{aligned} I &= \int \underbrace{\cos x}_{f'} \times \underbrace{\cos x}_{g} dx \\ &= \int \underbrace{(\sin x)'}_{f'} \times \underbrace{\cos x}_{g} dx \\ &= \sin x\cos x - \int \underbrace{\sin x}_{f} \times \underbrace{(\cos x)'}_{g'} dx \\ &= \sin x\cos x + \int \sin^2 x dx \\ &= \sin x\cos x + \int (1-\cos^2 x)dx \\ &= \sin x\cos x + x - I \\ I &= \frac{1}{2}(x + \sin x\cos x) + C \end{aligned}$$