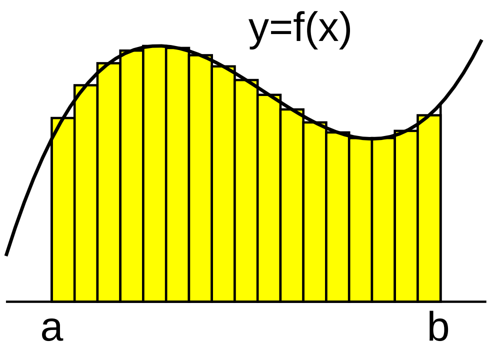
2 定積分

$$\lim_{\Delta \to 0} \sum_{k=1}^{N} f(\xi_k)(x_k - x_{k-1}) = \int_a^b f(x)dx$$

を $f(x)$ の定積分と呼ぶ。ただし、

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$$
$$x_{k-1} \le \xi_k \le x_k$$
$$|x_k - x_{k-1}| < \Delta, \qquad (k = 1, 2, 3, \cdots, N)$$

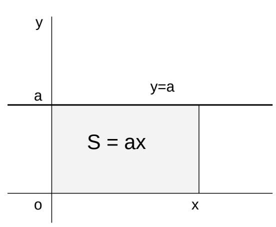
である。定積分は、曲線 $y = f(x)$、x 軸、$y = a$ および $y = b$ で囲まれた領域の符号を考慮した面積に等しい。

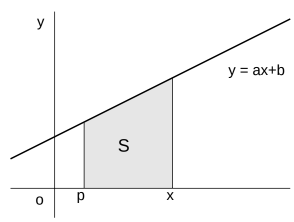


[例]

$$f_1(x) = a, \qquad \int_0^x f_1(x)dx = ax$$

$$f_2(x) = ax+b, \qquad \int_p^x f_2(x)dx = \frac{1}{2}ax^2+bx-(\frac{1}{2}ap^2+bp)$$





2.1 積分と微分の関係

積分と微分は互いに逆の演算である。

$$F(x) = \int f(x)dx + C \quad \longleftrightarrow \quad \frac{dF(x)}{dx} = f(x)$$

関数 $f(x)$ に対して、$F(x)$ を定義する:

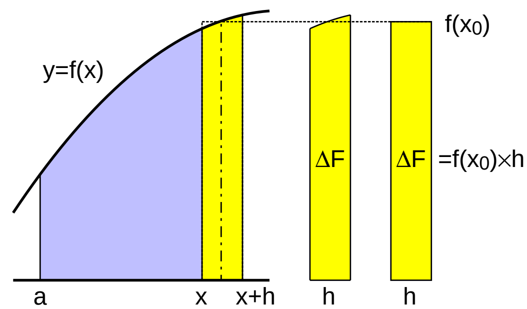
$$F(x) = \int_a^x f(t)dt$$

このとき、

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \\ &= f(x_0)h, \quad (x \le x_0 \le x+h) \end{aligned}$$

であるから、

$$\begin{aligned} \frac{d}{dx}F(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} f(x_0), \qquad (x \le x_0 \le x+h) \\ &= f(x) \end{aligned}$$



2.2 主な関数の積分

- $\int x^p dx = \frac{1}{p+1}x^{p+1} + C, \quad (p \neq -1)$
- $\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} + C$
- $\int \frac{1}{x}dx = \ln|x| + C$
- $\int \cos\omega t dt = \frac{1}{\omega}\sin\omega t + C$
- $\int \sin\omega t dt = -\frac{1}{\omega}\cos t + C$
- $\int e^{at}dt = \frac{1}{a}e^{at} + C$
- $\int \frac{1}{x}dx = \log x + C$

C: 積分定数

2.3 置換積分

$$\int f(x)dx = \int f(g(t))\frac{dx}{dt}dt = \int f(g(t))g'(t)dt$$

ただし、$x = g(t)$

$$F = \int f(x)dx$$

と表すと、

$$\frac{dF}{dt} = \frac{dF}{dx} \times \frac{dx}{dt} = f(x) \times g'(t)$$

両辺を t で積分して、

$$F = \int f(x)g'(t)dt = \int f(g(t))g'(t)dt$$

[例 1]

$$I = \int (2x+1)^2 dx$$

$t = 2x+1,\ x = \dfrac{t-1}{2}$ と置き換えると、

$$\begin{aligned} I &= \int t^2 \times \frac{dx}{dt} dx = \int t^2 \left(\frac{t-1}{2}\right)' dt \\ &= \frac{1}{2}\int t^2 dt = \frac{1}{6}t^3 + C \\ &= \frac{1}{6}(2x+1)^3 + C \end{aligned}$$

[例 2]

$$I = \int \sin 3x dx$$

$t = 3x$ と置き換えて、

$$\begin{aligned} I &= \int (\sin t) \times \frac{d}{dt}\left(\frac{t}{3}\right) dt \\ &= \frac{1}{3}\int \sin t dt = -\frac{1}{3}\cos t + C \\ &= -\frac{1}{3}\cos 3x + C \end{aligned}$$

$$\int f(g(x))g'(x)dx = \int f(t)dt$$

$t = g(x)$ と置き換えると、

$$\frac{dt}{dx} = g'(x)$$

であるから、

$$f(g(x))g'(x) = f(t)\frac{dt}{dx}$$

したがって、

$$\int f(g(x))g'(x)dx = \int f(t)\frac{dt}{dx}dx = \int f(t)dt$$

[例 3]

$$I = \int \frac{2x}{x^2+1}dx$$

$t = x^2 + 1$ と置き換えると、

$$dt = 2xdx$$

であるから、

$$\begin{aligned} I &= \frac{2xdx}{x^2+1} = \int \frac{dt}{t} \\ &= \log t + C = \log(x^2+1) + C \end{aligned}$$

2.4 部分積分

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx + C$$

関数の積の微分

$$\frac{d}{dx}\{f(x)g(x)\} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

の両辺を積分する。積分は微分の逆演算であるから、

$$f(x)g(x) + C = \int\{\frac{df(x)}{dx}g(x)\}dx + \int\{f(x)\frac{dg(x)}{dx}\}dx$$

$$\int\{\frac{df(x)}{dx}g(x)\}dx = f(x)g(x) - \int\{f(x)\frac{dg(x)}{dx}\}dx + C$$

が成立する。

[例]

$$\begin{aligned} \int xe^{-x}dx &= -\int x\left(\frac{d}{dx}e^{-x}\right)dx \\ &= -xe^{-x} + \int\left(\frac{d}{dx}x\right)e^{-x}dx \\ &= -xe^{-x} + \int e^{-x}dx \\ &= -(x+1)e^{-x} + C \end{aligned}$$

$$\begin{aligned} \int \log x dx &= \int 1 \cdot \log x dx = \int (x)' \log x dx \\ &= x\log x - \int x(\log x)' dx \\ &= x\log x - \int x \cdot \frac{1}{x}dx \\ &= x\log x - x + C \end{aligned}$$

2.5 まとめ

微分と積分は互いに逆演算

$$\frac{dF(x)}{dx} = f(x) \quad \Leftrightarrow \quad F(x) = \int f(x)dx + C$$

C は積分定数 (任意の定数)

置換積分

$$\int f\left(g(x)\right)\frac{dg(x)}{dx}dx = \int f(t)dt$$

部分分数展開を利用した積分

$$\begin{aligned}
&\int \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}}{(x-b_1)(x-b_2)\cdots(x-b_n)}dx \\
&= \int \left\{\frac{A_1}{x-b_1} + \frac{A_2}{x-b_2} + \cdots + \frac{A_n}{x-b_n}\right\}dx \\
&= A_1\log(x-b_1) + A_2\log(x-b_2) \\
&\qquad + \cdots + A_n\log(x-b_n) + C \\
&= \log\left\{(x-b_1)^{A_1}(x-b_2)^{A_2}\cdots(x-b_n)^{A_n}\right\} + C
\end{aligned}$$

ただし、

$$\begin{aligned}
A_0 &= \left.\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}}{(x-b_2)(x-b_3)\cdots(x-b_n)}\right|_{x=b_1} \\
A_1 &= \left.\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}}{(x-b_1)(x-b_3)(x-b_4)\cdots(x-b_n)}\right|_{x=b_2} \\
A_2 &= \left.\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}}{(x-b_1)(x-b_2)(x-b_4)\cdots(x-b_n)}\right|_{x=b_3} \\
\cdots & \quad \cdots \\
A_n &= \left.\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}}{(x-b_1)(x-b_2)\cdots(x-b_{n-2})(x-b_{n-1})}\right|_{x=b_n}
\end{aligned}$$

部分積分

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

2.5.1 例

置換積分

1. $I = \int x\sqrt{x^2+a}dx, \qquad (t = x^2 + a)$
2. $I = \int \frac{(1+\log x)^2}{x}dx, \qquad (t = 1 + \log x)$
3. $I = \int (1-\sin x)^3\cos x dx, \qquad (t = 1 - \sin x)$
4. $I = \int xe^{-x^2}dx, \qquad (t = -x^2)$
5. $I = \int e^x\cos(1+e^x)dx, \qquad (t = 1 + e^x)$

部分分数展開

1. $I = \int \frac{x^2+x+1}{x+2}dx = \int\left\{x - 1 + \frac{3}{x+2}\right\}dx$
2. $I = \int \frac{1}{(x+3)(x+4)}dx = \int\left\{\frac{A}{x+3} + \frac{B}{x+4}\right\}dx$

$$\begin{aligned}
A &= \left.\frac{1}{x+4}\right|_{x=-3} = 1 \\
B &= \left.\frac{1}{x+3}\right|_{x=-4} = -1
\end{aligned}$$

3. $I = \int \frac{x-7}{x^2-2x-3}dx = \int\left\{\frac{A}{x+1} + \frac{B}{x-3}\right\}dx$

$$\begin{aligned}
A &= \left.\frac{x-7}{x-3}\right|_{x=-1} \\
B &= \left.\frac{x-7}{x+1}\right|_{x=3}
\end{aligned}$$

4. $I = \int \frac{2x+3}{x(x+2)(x+3)}dx$

$$\frac{2x+3}{x(x+2)(x+3)} = \frac{a}{x} + \frac{b}{x+2} + \frac{c}{x+3}$$

$$\begin{aligned}
a &= \left.\frac{2x+3}{(x+2)(x+3)}\right|_{x=0} \\
b &= \left.\frac{2x+3}{x(x+3)}\right|_{x=-2} \\
c &= \left.\frac{2x+3}{x(x+2)}\right|_{x=-3}
\end{aligned}$$

部分積分

1. $I = \int xe^x dx, \quad (f = x,\ g' = e^x)$
2. $I = \int x\sin x dx, \quad (f = x,\ g' = \sin x)$
3. $I = \int x^2\log x dx, \quad (f = \log x,\ g' = x^2)$
4. $I = \int \sin 2x\cos x dx, \quad (f = \sin 2x,\ g' = \cos x)$

$$\begin{aligned}
I &= \underbrace{\sin 2x}_{f}\underbrace{\sin x}_{g} - \int \underbrace{(\sin 2x)'}_{f'}\underbrace{\sin x}_{g}dx \\
&= \sin 2x\sin x - 2\int \underbrace{\cos 2x}_{f_2}\underbrace{\sin x}_{g_2{}'}dx \\
&= \sin 2x\sin x \\
&\qquad +2\cos 2x\cos x - 2\int \underbrace{(-2\sin 2x)}_{f_2{}'}\cos x dx \\
&= \sin 2x\sin x + 2\cos 2x\cos x + 4I \\
I &= -\frac{1}{3}\{\sin 2x\sin x + 2\cos 2x\cos x\} + C
\end{aligned}$$